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KINETIC EQUATION FOR ONE-DIMENSIONAL MOTION OF SPHERES

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A dynamic system of identical spheres in a vessel is considered as a gas model. The spheres and the walls of the vessel are assumed to be absolutely rigid and elastic. For three-dimensional motion, the chain of Bogolyubov equations is derived in a form not available in the literature. It is shown that the reason for the noninvertibility of the Boltzmann kinetic equation is an approximate description of the dynamic system. For the one-dimensional motion of the spheres along the straight-line segment between the walls, the Bogolyubov chain is closed in the class of multiplicative distributions by a limiting transition in which the number of spheres tends to infinity and the sum of their diameters remains constant. The obtained kinetic equation differs substantially in structure from the Boltzmann equation. For example, it is invertible. It implies the equations of multivelocity hydrodynamics. The existence of the solution in the large in time is established. It is shown that a closed kinetic equation for the one-particle projection does not exist in the class of arbitrary distributions.

Key words: *ergodicity, noninvertibility, s-particle projection, kinetic equation, s-multiplicative distribution, crystallographic group, hydrodynamics.*

1. Ergodicity and Noninvertibility. The model of molecules as absolutely rigid elastic spheres has played an important role in the kinetic theory of gases. The problem of the derivation of the kinetic equation was discussed in the beginning of the 20th century. When Boltzmann derived his famous kinetic equation for this system, the equation turned out to be invertible, i.e., invariant under a change in the time direction, although the starting system of equations of motion for the spheres and the laws of their collisions are invertible [1]. This inconsistency received strong criticism at that time. Analyzing his derivation, Boltzmann had to add an additional molecular chaos hypothesis to the dynamic system considered. Subsequently, Bogolyubov gave the following mathematical formulation of this hypothesis [2].

Bogolyubov considered the Liouville equation for the probability density $w(t, \boldsymbol{x}_1, \boldsymbol{v}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{v}_N)$ (t is time; $\boldsymbol{x}_k$ and $\boldsymbol{v}_k$ are the coordinates of the center and velocity of the kth sphere). On the boundary of the configuration space of the points $x = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ — cylinders of the form $|\boldsymbol{x}_i - \boldsymbol{x}_j| = d_1$ (d_1 is the diameter of the spheres; $|\boldsymbol{x}_k|$ is the length of the vector $\boldsymbol{x}_k \in \mathbb{R}^3$; $1 \le i < j \le N$) — and on the vessel walls, the function w satisfies the boundary conditions that follow from the collision laws. In the space of the velocities $v = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N)$, boundaries are absent.

The boundary problem for the function w is equivalent to the initial dynamic system but has a number of advantages. First, because the spheres are considered identical, the initial function $w(0, x, v)$ can be chosen to be symmetric, i.e., constant under an arbitrary permutation of its variables, which that corresponds to a permutation of the spheres. Then, this property is preserved for $t > 0$. Second, we can consider the s-particle projections obtained by integrating the probability density over the phase space of $N - s$ particles:

$$w_s(t, \boldsymbol{x}_1, \boldsymbol{v}_1, \ldots, \boldsymbol{x}_s, \boldsymbol{v}_s) = \int w(t, \boldsymbol{x}_1, \boldsymbol{v}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{v}_N)\, d\boldsymbol{x}_{s+1}\, d\boldsymbol{v}_{s+1} \cdots d\boldsymbol{x}_N\, d\boldsymbol{v}_N$$

$$(s = 1, 2, \ldots).$$

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If s is small, these functions are physically measurable. For example, Nw_1 has the meaning of the average particle density in the phase space of one particle. The Boltzmann kinetic equation is written for the one-particle projection w_1.

For definiteness, let external force fields be absent. Then, the system of equations of motion for the spheres is written as

$$\dot{\boldsymbol{x}}_i = \boldsymbol{v}_i, \qquad \dot{\boldsymbol{v}}_i = 0 \qquad (i = 1, \ldots, N), \tag{1.1}$$

where the dot above denotes the derivative with respect to time. We denote by $\Omega' \subset \mathbb{R}^3$ the vessel which contain N identical spheres, and by $\Omega \subset \Omega'$ the closed subset separated from the vessel walls by the distance $d_1/2$. The vessel walls $\partial\Omega'$ are assumed to be a smooth surface such that a sphere of diameter d_1 touches it at only one point. Then, for each point $\boldsymbol{x}$ of the boundaries $\partial\Omega$ of the domain Ω, there is a single nearest point $\boldsymbol{x}' \in \partial\Omega'$ and the vector $\boldsymbol{n} = (\boldsymbol{x}' - \boldsymbol{x})2d_1^{-1}$ is the common outward normal to $\partial\Omega$ and $\partial\Omega'$ at the points $\boldsymbol{x}$ and $\boldsymbol{x}'$. The vessel walls, as well as the spheres, are assumed to be absolutely rigid and elastic. The centers of the spheres $\boldsymbol{x}_i$ can be at any point from Ω. The phase space of system (1.1) is equal to $\Phi = Q \times \mathbb{R}^{3N}$, where

$$Q = \Omega^N \setminus \Big(\bigcup_{1 \le i < j \le N} \{x\colon |\boldsymbol{x}_i - \boldsymbol{x}_j| < d_1\} \Big).$$

The reflection of the trajectory $[x(t), v(t)]$, where $0 \le t < \infty$, of system (1.1) from the boundary $\partial\Phi$ of the phase space Φ is determined by the collision laws. As a result of collision, the coordinate $x \in \partial Q$ does not change and the velocity $v \in \mathbb{R}^{3N}$ suddenly takes a new value v':

$$\boldsymbol{v}_i' = \boldsymbol{v}_i - 2(\boldsymbol{v}_i \cdot \boldsymbol{n})\boldsymbol{n} \quad \text{if} \quad \boldsymbol{x}_i \in \partial\Omega \text{ and } \boldsymbol{v}_i \cdot \boldsymbol{n} > 0; \tag{1.2}$$

$$\boldsymbol{v}_i' = \boldsymbol{v}_i - (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e}, \quad \boldsymbol{v}_j' = \boldsymbol{v}_j + (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e} \quad \text{if} \quad |\boldsymbol{x}_i - \boldsymbol{x}_j| = d_1 \text{ and } \boldsymbol{v} \cdot \boldsymbol{e} > 0. \tag{1.3}$$

Here $\boldsymbol{n}$ is the outward normal to $\partial\Omega$, $\boldsymbol{e} = (\boldsymbol{x}_j - \boldsymbol{x}_i)d_1^{-1}$, and $\boldsymbol{v} = \boldsymbol{v}_i - \boldsymbol{v}_j$. These are pair collisions. The set of trajectories of measure 0 undergoes higher-order collisions and is discarded.

We write the Liouville equation

$$\frac{\partial w}{\partial t} + \sum_{i=1}^{N} \boldsymbol{v}_i \cdot \nabla_i w = 0, \tag{1.4}$$

where ∇_i is the gradient over $\boldsymbol{x}_i$. The solution w of Eq. (1.4) has a constant value along the trajectory of system (1.1); therefore, as the boundary conditions, it is natural to assume that this value is retained for the reflection of the trajectory from the boundary: $w(t, x, v) = w(t, x, v')$ for $(x, v) \in \partial\Phi$. Because the inverse map $v' \to v$ is given by the same formulas (1.2) and (1.3), this equality is valid at all points of the boundary of the phase space and not only at the ends of the incoming trajectories.

The solution of the Liouville equation satisfying the indicated boundary condition is uniquely determined by its initial value, so that there arises a linear operator S^t: $w(0, x, v) \to w(t, x, v)$, which depends on the parameter t. It is easy to prove that the integral of w over Φ remains constant in time; therefore, the normalization of the probability density $\int\limits_{\Phi} w\, dx\, dv = 1$ is conserved. The last property also implies the adopted boundary conditions. Equation (1.4) and the boundary conditions are satisfied by a function of the form

$$w = f(E),$$

where $E = (|\boldsymbol{v}_1|^2 + \ldots + |\boldsymbol{v}_N|^2)/2$ is the energy of the system (if the mass of the sphere is set equal to unity) and f is an arbitrary function of one variable. If $w(0, x, v) = 0$ for $E \ge E_0$, then $w(t, x, v) = 0$ for $E \ge E_0$ at any time t.

To obtain the equation for the s-particle projection w_s, it is necessary to integrate Eq. (1.4) over the phase space of spheres with numbers $s + 1, \ldots, N$, i.e., over the cross section $\bar{\Phi}$ of the phase space Φ for fixed $\boldsymbol{x}_i$ and $\boldsymbol{v}_i$ $(1 \le i \le s)$. The s-particle projection w_s was defined above as the integral of w over $\bar{\Phi}$. The set $\bar{\Phi}$ has the form $\bar{Q} \times \mathbb{R}^{3(N-s)}$, and its boundary has the form $\partial\bar{Q} \times \mathbb{R}^{3(N-s)}$, where $\bar{Q}$ is the cross section of the configuration space Q for the fixed coordinates $\boldsymbol{x}_i$ $(1 \le i \le s)$. The boundary $\partial\bar{Q}$ of the cross section $\bar{Q}$ is the union of the nonintersecting (except for the set of measure 0) components equal to the intersections of the following surfaces with $\bar{Q}$:

— the walls

$$\bigcup_{i=s+1}^{N} \{x\colon \boldsymbol{x}_i \in \partial\Omega\};$$

— the cylinders

$$\bigcup_{s<i<j\le N} \{x\colon |\boldsymbol{x}_i - \boldsymbol{x}_j| = d_1\};$$

— the spheres

$$\bigcup_{i=1}^{s}\bigcup_{j=s+1}^{N} \{x\colon |\boldsymbol{x}_i - \boldsymbol{x}_j| = d_1\}.$$

The integral over $\partial\bar{Q}$ is equal to the sum of the integrals over the given components.

The time derivative can be factored outside the integral sign of the projection w_s. Let us consider the integral of the next s terms. Since only the boundary spheres depend on $\boldsymbol{x}_i$ $(1 \le i \le s)$, we have

$$\int\limits_{\bar\Phi} \boldsymbol{v}_i \cdot \nabla_i w \, d\bar{x}\, d\bar{v} = \boldsymbol{v}_i \cdot \nabla_i w_s - \sum_{j=s+1}^{N} \int\limits_{|\boldsymbol{x}_i - \boldsymbol{x}_j| = d_1} \boldsymbol{v}_i \cdot (\boldsymbol{x}_i - \boldsymbol{x}_j) d_1^{-1} \, d\bar{x}\, d\bar{v}. \tag{1.4′}$$

Here $d\bar{x}\, d\bar{v} = d\boldsymbol{x}_{s+1}\, d\boldsymbol{v}_{s+1} \cdots d\boldsymbol{x}_N\, d\boldsymbol{v}_N$; in the second part of the equality, the integration is performed over the set

$$(\{x\colon |\boldsymbol{x}_i - \boldsymbol{x}_j| = d_1\} \cap \bar{Q}) \times \mathbb{R}^{3(N-s)},$$

$d\boldsymbol{x}_j$ denotes the differential of the surface area of the sphere $|\boldsymbol{x}_i - \boldsymbol{x}_j| = d_1$.

According to the Gauss–Ostrogradskii formula, the integral of the remaining terms of the Liouville equation is transformed to the surface integral of $(\boldsymbol{v}_{s+1} \cdot \boldsymbol{\nu}_{s+1} + \ldots + \boldsymbol{v}_N \cdot \boldsymbol{\nu}_N)w$ over $\partial\bar{\Phi}$, where $(\boldsymbol{\nu}_{s+1}, \ldots, \boldsymbol{\nu}_N)$ is the outward normal to $\partial\bar{Q}$.

Let us consider the integral over the wall $\{x\colon \boldsymbol{x}_i \in \partial\Omega\}$ $(s < i \le N)$. Because the outward normal to this surface has one nonzero component $\boldsymbol{\nu}_i = \boldsymbol{n}$, the function $\boldsymbol{v}_i \cdot \boldsymbol{n} w$ is integrated over the Cartesian product (the intersection with $\bar{Q}$) of this wall and $\mathbb{R}^{3(N-s)}$. For fixed $\bar{x} = (\boldsymbol{x}_{s+1}, \ldots, \boldsymbol{x}_N)$, we perform integration over $\bar{v} = (\boldsymbol{v}_{s+1}, \ldots, \boldsymbol{v}_N)$. The integration domain $\mathbb{R}^{3(N-s)}$ is divided into two subdomains by the inequalities $\boldsymbol{v}_i \cdot \boldsymbol{n} > 0$ and $\boldsymbol{v}_i \cdot \boldsymbol{n} < 0$. In the subdomain $\boldsymbol{v}_i \cdot \boldsymbol{n} > 0$, we make the change of the integration variable $\boldsymbol{v}_i \to \boldsymbol{v}_i'$ using the mapping (1.2). Because

$$\boldsymbol{v}_i \cdot \boldsymbol{n} = -\boldsymbol{v}_i' \cdot \boldsymbol{n}, \qquad d\boldsymbol{v}_i = d\boldsymbol{v}_i',$$

the result is an integral that differs from the integral over the remaining subdomain $\boldsymbol{v}_i \cdot \boldsymbol{n} < 0$ only in sign (and the notation of the integration variable) by virtue of the boundary conditions on w. Thus, the integrals over the walls are equal to zero.

Next, the outward normal to the cylinder $\{x\colon |\boldsymbol{x}_i - \boldsymbol{x}_j| = d_1\}$ $(s < i < j \le N)$ has only two nonzero components: $\boldsymbol{\nu}_i = -(\boldsymbol{x}_i - \boldsymbol{x}_j)d_1^{-1}/\sqrt{2}$ and $\boldsymbol{\nu}_j = -\boldsymbol{\nu}_i$. Therefore, we integrate the function

$$(\boldsymbol{v}_i - \boldsymbol{v}_j) \cdot ((\boldsymbol{x}_j - \boldsymbol{x}_i)d_1^{-1}/\sqrt{2})w.$$

The domain of integration $\mathbb{R}^{3(N-s)}$ over $\bar{v}$ is again divided into two subdomains by the inequality

$$(\boldsymbol{v}_i - \boldsymbol{v}_j) \cdot (\boldsymbol{x}_j - \boldsymbol{x}_i) > 0$$

and the opposite inequality. In the integral over the first subdomain, we make the change of integration variables (1.3). By virtue of the equalities

$$(\boldsymbol{v}_i - \boldsymbol{v}_j) \cdot (\boldsymbol{x}_j - \boldsymbol{x}_i) = -(\boldsymbol{v}_i' - \boldsymbol{v}_j') \cdot (\boldsymbol{x}_j - \boldsymbol{x}_i), \qquad d\boldsymbol{v}_i d\boldsymbol{v}_j = d\boldsymbol{v}_i' d\boldsymbol{v}_j'$$

and the boundary conditions $w(t, x, v) = w(t, x, v')$, this integral and the integral in the second subdomain cancel each other. Thus, the integrals over the cylinders are also equal to zero.

The integrals over the spheres are retained. The normal to the sphere $\{x\colon |\boldsymbol{x}_i - \boldsymbol{x}_j| = d_1\}$ $(1 \le i \le s < j \le N)$ has only one nonzero component $\boldsymbol{\nu}_j = (\boldsymbol{x}_i - \boldsymbol{x}_j)d_1^{-1}$; therefore, the function $\boldsymbol{v}_j \cdot (\boldsymbol{x}_i - \boldsymbol{x}_j)d_1^{-1} w$ is integrated. Summing these integrals over j and adding the result to the integral (1.4′), we see that the expression under the summation

sign does not depend on the value of the subscript j because of the symmetry of the probability density w and the set $\bar{Q}$. Therefore, we can set $j = s+1$ and replace the sum over j by the factor $N-s$. We finally obtain

$$\frac{\partial w_s}{\partial t} + \sum_{i=1}^{s} \boldsymbol{v}_i \cdot \nabla_i w_s = J_{s+1} \qquad (s = 1, 2, \ldots, N), \tag{1.5}$$

where $w_N = w$ and $J_{N+1} = 0$; for $s < N$, we have

$$J_{s+1} = (N-s) \sum_{i=1}^{s} \int (\boldsymbol{v}_i - \boldsymbol{v}_{s+1}) \cdot (\boldsymbol{x}_i - \boldsymbol{x}_{s+1})$$

$$\times\, d_1^{-1} w_{s+1}(t, \boldsymbol{x}_1, \boldsymbol{v}_1, \ldots, \boldsymbol{x}_{s+1}, \boldsymbol{v}_{s+1})\, d\boldsymbol{x}_{s+1}\, d\boldsymbol{v}_{s+1}. \tag{1.6}$$

Here the integration over $\boldsymbol{x}_{s+1}$ is performed over the segments of the spheres $|\boldsymbol{x}_{s+1} - \boldsymbol{x}_i| = d_1$ $(i = 1, \ldots, s)$ that lie in Ω and do not overlap each other ($d\boldsymbol{x}_{s+1}$ is the differential of the surface area of the sphere), and the integration over $\boldsymbol{v}_{s+1}$ is performed over the space $\mathbb{R}^3$. The function w_s has a range of definition and satisfies the boundary conditions as the probability density of the dynamic system of s spheres. Indeed, for the projection w_s, the conditions $\boldsymbol{x}_i \in \Omega$ $(1 \le i \le s)$ and the inequalities $|\boldsymbol{x}_i - \boldsymbol{x}_j| \ge d_1$ $(1 \le i < j \le s)$ remain meaningful. The subset $\Psi \subset \bar{\Phi}$ of measure 0 corresponds to collisions with the participation of the jth sphere $(j > s)$; therefore,

$$w_s(t, \boldsymbol{x}_1, \boldsymbol{v}_1, \ldots, \boldsymbol{x}_s, \boldsymbol{v}_s) = \int\limits_{\bar{\Phi}\setminus\Psi} w(t, x, v)\, d\bar{x}\, d\bar{v}.$$

Setting $w(t, x, v) = w(t, x, v')$ on the right of this equality, we obtain the boundary condition on w_s that corresponds to the collisions (1.2) and (1.3) involving only spheres with numbers $1, \ldots, s$. The statement is proved. The chain of Bogolyubov boundary-value problems (1.5), (1.6) equivalent to the initial dynamic system (1.1)–(1.3) and, hence, it is invertible, i.e., invariant under the change of variables $t \to -t$ and $v \to -v$.

The chain of Bogolyubov equations is usually derived for the case of paired potential interaction of particles. The collision integral in the form (1.6) that is not encountered in the literature.

Let us consider the collision integral J_2 in more detail. We make the change of the integration variable $\boldsymbol{x}_2 \to \boldsymbol{e} = (\boldsymbol{x}_2 - \boldsymbol{x}_1) d_1^{-1}$, where $|\boldsymbol{e}| = 1$ and $d\boldsymbol{x}_2 = d_1^2 d\boldsymbol{e}$. Next, we divide the domain of integration $\mathbb{R}^3$ over $\boldsymbol{v}_2$ into two subdomains by the inequality $(\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \boldsymbol{e} > 0$ and the opposite inequality. In the second subdomain, we make the change of the integration variable $\boldsymbol{e} \to -\boldsymbol{e}$ and, according to the boundary conditions, we set

$$w_2(t, \boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{x}_1 - \boldsymbol{e}d_1, \boldsymbol{v}_2) = w_2(t, \boldsymbol{x}_1, \boldsymbol{v}_1', \boldsymbol{x}_1 - \boldsymbol{e}d_1, \boldsymbol{v}_2'),$$

where $(\boldsymbol{v}_1, \boldsymbol{v}_2) \to (\boldsymbol{v}_1', \boldsymbol{v}_2')$ is the mapping (1.3). To obtain the Boltzmann kinetic equation, it is necessary to assume the equality

$$w_2(t, \boldsymbol{x}_1, \boldsymbol{v}_1', \boldsymbol{x}_1 - \boldsymbol{e}d_1, \boldsymbol{v}_2') - w_2(t, \boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{x}_1 + \boldsymbol{e}d_1, \boldsymbol{v}_2)$$

$$\simeq w_1(t, \boldsymbol{x}_1, \boldsymbol{v}_1') w_1(t, \boldsymbol{x}_1, \boldsymbol{v}_2') - w_1(t, \boldsymbol{x}_1, \boldsymbol{v}_1) w_1(t, \boldsymbol{x}_1, \boldsymbol{v}_2) \tag{1.7}$$

$$\text{at} \quad (\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \boldsymbol{e} > 0.$$

Consequently, this is the molecular chaos hypothesis. Then Eq. (1.5) for $s = 1$ takes the classical form

$$\frac{\partial w_1}{\partial t} + \boldsymbol{v}_1 \cdot \nabla_1 w_1 = (N-1) d_1^2 \int\limits_{(\boldsymbol{v}_1 - \boldsymbol{v}_2)\cdot \boldsymbol{e} > 0} (\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \boldsymbol{e}$$

$$\times\, (w_1(t, \boldsymbol{x}_1, \boldsymbol{v}_1') w_1(t, \boldsymbol{x}_1, \boldsymbol{v}_2') - w_1(t, \boldsymbol{x}_1, \boldsymbol{v}_1) w_1(t, \boldsymbol{x}_1, \boldsymbol{v}_2))\, d\boldsymbol{e}\, d\boldsymbol{v}_2. \tag{1.8}$$

In Eq. (1.8), the changes $t \to -t$, $\boldsymbol{v}_1 \to -\boldsymbol{v}_1$, and $\boldsymbol{v}_2 \to -\boldsymbol{v}_2$ are no longer valid since it changes the integration domain or, in the case of the simultaneous change $\boldsymbol{e} \to -\boldsymbol{e}$, the left and right sides of the equation acquire different factors ∓ 1. Thus, Eq. (1.8) is noninvertible. The noninvertibility arises because of the molecular chaos hypothesis (1.7).

The noninvertibility paradox of the Boltzmann kinetic equation was solved by P. A. Ehrenfest and T. A. Ehrenfest-Afanassjewa, who understood that the one-particle projection w_1 does not uniquely characterize the state of the dynamic system or the complete function w [3]. Therefore, the function w_1 evolves to a more

probable value because of the uncontrolled changes in the dynamic system. From this it also follows that the deterministic evolution of w_1 is possible only in a narrow class of distributions w. Bogolyubov derives the kinetic equation for w_1 in the class of distributions w that possess the following property: each s-particle projection w_s is uniquely determined by the one-particle projection w_1. However, he does not give any example of a distribution w that belongs or does not belong to this class [2].

Sinai and Chernov [4] proved ergodicity for the dynamic system (1.1)–(1.3). The physical meaning of ergodicity is as follows: irrespective of the initial state of a system with energy E_0, the average time of residence of the system in each neighborhood on the constant-energy surface $E = E_0$ is identical if these neighborhoods have the same area. The mathematical formulation is as follows: for any initial distribution w_0, there exists a function of one variable f such that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t (S^\tau w_0)\, d\tau = f\Big(\frac{|\boldsymbol{v}_1|^2}{2} + \ldots + \frac{|\boldsymbol{v}_N|^2}{2}\Big). \tag{1.9}$$

This statement is proved in an appropriate functional space. Does ergodicity leads to the molecular chaos hypothesis or noninvertibility? No, it does not. Letting $t \to -\infty$ on the left of equality (1.9), we obtain the same function on the right side. Thus, ergodicity does not distinguish one predominant direction of the two time directions.

2. Example of Noninvertibility. As shown in Sec. 1, the reason for the noninvertibility of the Boltzmann kinetic equation is the approximation (1.7), i.e., an inaccurate, incomplete description of the dynamic system. At the same time, a full description is meaningless since it cannot be verified experimentally. This point of view on the nature of physical noninvertibility agrees with [3]. It is vividly illustrated by the following simple example.

In the Hilbert space L_2 of quadratically integrable periodic functions with period 1, we define a unitary operator U by the formula

$$(Uf)(x) = \exp(2\pi i x) f(x).$$

We consider the semigroup of operators $\{U^n\}$ with a discrete parameter $n = 0, 1, 2, \ldots$ as an analog of the one-parameter semigroup $\{S^t\}$ (see Sec. 1). For any functions $f, g \in L_2$, we have

$$(U^n f, g) = \int_0^1 \exp(2\pi i n x) f(x) g(x)^*\, dx \underset{n\to\infty}{\longrightarrow} 0, \tag{2.1}$$

where $(\,\cdot\,,\cdot\,)$ is the scalar product in L_2; the asterisk denotes complex conjugation. Because the function $f(x)g(x)^*$ is absolutely integrable, this statement follows from the Riemann theorem. Is it possible to state for this reason that the dynamic system $\{U^n\}$ is noninvertible? No, it is not because by letting $n \to -\infty$, we obtain the same zero in the limit.

We assume that only the average value of the function $f \in L_2$ is known:

$$\bar f(x) = (Pf)(x) = \int_{-\infty}^{\infty} (2\pi\chi)^{-1/2} \exp\Big(-\frac{(x-y)^2}{2\chi}\Big) f(y)\, dy.$$

Here $\chi > 0$ is a small but fixed constant. Any measuring instrument performs similar averaging. The averaging operator P is continuous and biunique but the inverse operator P^{-1} is unbounded and is not defined everywhere. We have the mapping

$$\bar f = Pf \to PUf = (PUP^{-1})Pf = (PUP^{-1})\bar f.$$

From this it follows that the evolution of the average quantities $\overline{U^n f}$ is defined by the operator $U' = PUP^{-1}$ and, thus, a new semigroup $\{U'^n\}$ arises.

To find the range of definition of the operator U', we expand the function $f \in L_2$ in a Fourier series:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \exp(2\pi i k x).$$

Then, the functions Uf and Pf have the kth Fourier coefficients c_{k-1} and $b^{-k^2} c_k$, respectively, where $b = \exp(8\pi^2\chi) > 1$. From this, using the Parseval formula, we obtain the following condition for the fact that $U'^n f \in L_2$:

$$\|U'^n f\|^2 = \sum_{k=-\infty}^{\infty} |b^{-2nk-n^2} c_k|^2 < \infty. \tag{2.2}$$

Consequently, the semigroup $\{U'^n\}$ is defined only for functions $f \in L_2$ whose Fourier coefficients satisfy inequality (2.2) for any positive number n, i.e., for $k \to -\infty$, they decrease faster than any exponent, for example, as b^{-k^2}. We denote the subset of such functions by D_+. This semigroup is noninvertible because the inverse operator U'^{-1} is not even defined on the entire subset $D_+ \subset L_2$. Thus, we obtained the noninvertible system $\{U'^n\}$ proceeding from the invertible system $\{U^n\}$. Noninvertibility was introduced by the averaging operator P and the subset of functions $D_+ \subset L_2$. It is obvious that there is an analogy between the operator P and the projection $w \to w_1$ and between the set D_+ and the Bogolyubov class of distributions w.

It is also possible to consider the subset of functions $D_- \subset L_2$ whose Fourier coefficients decrease faster than the exponent as $k \to \infty$. Then, inequality (2.2) holds for any $n \le 0$. Consequently, a noninvertible semigroup that evolves into the "past" as $n \to -\infty$ is defined on the subset D_-. Thus, the predominant time direction n is determined by a choice of the subset D_+ or D_-. On the subset $D_+ \cap D_-$, the semigroup $\{U'^n\}$ is invertible in the sense that both directions of n are equivalent.

3. One-Dimensional Motion. Change of Coordinates $x \to y$. In the one-dimensional case, we denote the coordinates of the centers and velocities of the spheres by $x = (x_1, \ldots, x_N)$ and $v = (v_1, \ldots, v_N)$, respectively. The spheres are enumerated from left to right, so that the following inequalities hold:

$$0 \le x_1 \le x_2 - d_1 \le \ldots \le x_N - (N-1)d_1 \le 1. \tag{3.1}$$

Denote this simplex by D_x and the phase space $D_x \times \mathbb{R}^N$ by Φ_x. Thus, the centers of the spheres are within the segment $[0, a]$ $(a = 1 + (N-1)d_1)$, which in this case plays the role of the set Ω. The collision laws are strongly simplified. Upon reflection from the ends of the segment $[0, a]$, the spheres acquire opposite velocities, and during collision with each other, they interchange velocities. In the Liouville equation (1.4), $\boldsymbol{v}_i \cdot \nabla_i$ is replaced by $v_i \partial/\partial x_i$.

However, if the probability density $w(t, x, v)$ is specified only in the simplex (3.1), the s-particle projections cannot be determined. Therefore, we continue the function w to the set $[0, a]^N \times \mathbb{R}^N$ by virtue of the symmetry property. Let S be the group of permutations of the numbers $1, 2, \ldots, N$. If $\sigma \in S$, we denote by σ the following linear maps:

$$x \to (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(N)}), \quad (x, v) \to (\sigma x, \sigma v), \quad f(x, v) \to f(\sigma^{-1} x, \sigma^{-1} v).$$

The set of simplices σD_x $(\sigma \in S)$ covers the cube $[0, a]^N$, except for the slots $|x_i - x_j| < d_1$ $(1 \le i < j \le N)$, in which we set $w = 0$ for any $v \in \mathbb{R}^N$. On the set $\sigma\Phi_x$, we define the probability density by the equality $w = \sigma(w|_{\Phi_x})$. Here $w\Big|_{\Phi_x}$ is the narrowing of the function w on set Φ_x.

We make the change of coordinates:

$$x_i \to y_i = x_i - (i-1)d_1 \qquad (i = 1, \ldots, N).$$

In this case, the simplex D_x becomes D:

$$0 \le y_1 \le y_2 \le \ldots \le y_N \le 1. \tag{3.2}$$

The simplices σD $(\sigma \in S)$ cover the unit cube $Q_1 = [0, 1]^N$ without slots and overlaps. The larger cube $Q_2 = [-1, 1]^N$ is similarly covered by reflections Q_1 in the coordinate planes. Thus, the group of reflections R arises. An element $\rho \in R$ acts by the formula $\rho y = (\chi_1 y_1, \ldots, \chi_N y_N)$, where $\chi_i = \pm 1$ $(1 \le i \le N)$. Finally, by shifting the cube Q_2 by vectors with even integer coordinates, it is possible to cover the entire space $\mathbb{R}^N$. We denote the corresponding group of translations by T. The groups R and T are commutable and are permutable with each element from S and RS, respectively. The reflection $\rho \in R$ and translation $\tau \in T$ by the vector $\bar{\tau}$, as well as the permutation, generate linear operators

$$(\rho f)(y, v) = f(\rho^{-1} y, \rho^{-1} v), \qquad (\tau f)(y, v) = f(y - \bar{\tau}, v).$$

In contrast to the permutations and reflections, the translations do not change the velocity.

Let $w'(t, y, v) = w(t, x, v)$ be a solution of the Liouville equation in the phase space $\Phi = D \times \mathbb{R}^N$ that satisfies the boundary conditions. Continuing it with the aid of the group S $[w' = \sigma(w'|_\Phi)$ to $\sigma\Phi$ $(\sigma \in S)]$, we obtain the solution in the set $\Phi_1 = Q_1 \times \mathbb{R}^N$.

Next, we continue w' from Φ_1 to $\Phi_2 = Q_2 \times \mathbb{R}^N$ using the group R. This continuation of $w'|_{\Phi_2}$ is invariant with respect to the group R by construction, but it will also be invariant with respect to the group S by virtue of the permutability of R with each element from S. Then, we continue w' from Φ_2 to $\mathbb{R}^{2N}$ using the group T. As a result, we have a solution of the Liouville equation in $\mathbb{R}^{2N}$ that is invariant with respect to the group TRS. If the initial value $w'_0(y,v)$ specified on Φ is continued to $\mathbb{R}^{2N}$ as is indicated above, the solution is represented by the explicit formula

$$w'(t,y,v) = w'_0(y - vt, v). \tag{3.3}$$

In the coordinates y, we have a system of seemingly noninteracting particles. Therefore, each s-particle projection is uniquely determined by the initial value.

4. Kinetic Equation. We express the one-particle projection w_1 in the coordinates x in terms of the symmetrical probability density w' in the coordinates y. By the definition, we have

$$w_1(t,x_1,v_1) = \int w(t,x,v)\,dx_2\,dv_2\,\cdots\,dx_N\,dv_N, \tag{4.1}$$

where the integration domain is $([0,a]\times\mathbb{R})^{N-1}$. The change of variables $x \to y$ is possible only in the simplices (3.1) and (3.2); therefore, we reduce the integral (4.1) to an integral over the subset of the set Φ_x using the symmetry of the function w.

We denote by Π a hyperplane with fixed x_1 and v_1. The integral (4.1) is taken over the variables x and v which belong to the union of subsets $\sigma\Phi_x \cap \Pi$ for all $\sigma \in S$. We map the subset $\sigma\Phi_x \cap \Pi$ to Φ_x using the permutation σ^{-1}, make the change of coordinates $x \to y$ and velocities $v \to v' = v$, and then map the result to $\sigma\Phi$ using the permutation σ. We have the integral of the function $w'(t,y,v') = w(t,x,v)$ over the set

$$\bigcup_{\sigma\in S} \sigma\Phi \cap \{y_1 = x_1 - (\sigma^{-1}(1)-1)d_1, v'_1 = v_1\}$$

$$= \bigcup_{i=1}^{N} \bigcup_{\alpha_1,\ldots,\alpha_{i-1}} \{0 \le y_{\alpha_1},\ldots,y_{\alpha_{i-1}} \le \eta \le y_{\alpha_i},\ldots,y_{\alpha_{N-1}} \le 1\} \times \{v'_1 = v_1\}, \tag{4.2}$$

where $y_1 = \eta = x_1 - (i-1)d_1$; $\{\alpha_1,\ldots,\alpha_{N-1}\} = M' = \{2,\ldots,N\}$; the union is taken over all subsets $\{\alpha_1,\ldots,\alpha_{i-1}\} \subset M'$, which contain $i-1$ elements. We note that here we first take the union over all σ that satisfy the condition $\sigma(i) = 1$ and then take the union over i.

We calculate the integral of the following function over the set (4.2):

$$w'(t,y,v') = w'_1(t,y_1,v'_1)w'_1(t,y_2,v'_2)\cdots w'_1(t,y_N,v'_N).$$

Since in the coordinates y, the spheres are seemingly noninteracting, this distribution means random, i.e., independent, motion of the spheres. However, this independence is not only a consequence of the absence of interaction but is also ensured by the special initial 1-multiplicative distribution. In Sec. 5, we consider more general s-multiplicative initial distributions. Then, the solution for $t > 0$ will also be s-multiplicative; therefore, the motion of the spheres is independent (in the coordinates y). In such cases, physicists do not confine themselves to a consideration of 1-multiplicative distributions because unaccounted small interactions are assumed to randomize the state of the system. Mathematicians attempt to consider the most general case.

The function w' is considered normalized; therefore, the integral of w'_1 over the set $[0,1]\times\mathbb{R}$ is equal to 1. We obtain

$$w_1(t,x_1,v_1) = \sum_{i=1}^{N} w'_1(t,\eta,v_1)\binom{N-1}{i-1}\Big(\int_0^\eta\int_{\mathbb{R}} w'_1(t,y_2,v'_2)\,dy_2\,dv'_2\Big)^{i-1}\Big(\int_\eta^1\int_{\mathbb{R}} w'_1(t,y_2,v'_2)\,dy_2\,dv'_2\Big)^{N-i}. \tag{4.3}$$

We let $N \to \infty$ for constants $z = (i-1)/(N-1)$ and $\varepsilon = (N-1)d_1$. The quantity $\eta = x_1 - \varepsilon z$ and, hence, and the integrals in formula (4.3) do not depend on N. If the first of these integrals is denoted by p, the second integral will be equal to $1-p$ $(0 < p < 1)$. Then, according to the limiting theorem of probability theory, the sum (4.3) converges to the integral

$$\int_0^1 w'_1(t,\eta,v_1)\delta(z-p)\,dz = \frac{w'_1(t,\eta,v_1)}{1 - dp/dz}\Big|_{z=p}$$

(see [5]). Thus,

$$w_1(t, x_1, v_1) = w_1'(t, y_1, v_1)(1 + \varepsilon \partial p(t, y_1)/\partial y_1)^{-1}, \tag{4.4}$$

where

$$x_1 = y_1 + \varepsilon p(t, y_1), \qquad p(t, y_1) = \int_0^{y_1} \int_{\mathbb{R}} w_1'(t, y_2, v_2')\, dy_2\, dv_2'.$$

Since there is a single-valued map $w_1'(0, y_1, v_1) \to w_1'(t, y_1, v_1)$ specified by formula (3.3), by virtue of equality (4.4), we have the semigroup

$$w_1(0, x_1, v_1) \to w_1(t, x_1, v_1).$$

The infinitesimal operator of this semigroup gives the kinetic equation for $w_1(t, x_1, v_1)$

$$\frac{\partial w_1}{\partial t} + \frac{\partial (u w_1)}{\partial x_1} = 0, \tag{4.5}$$

where

$$u = \Big(v_1 - \varepsilon \int_{\mathbb{R}} v_2 w_1(t, x_1, v_2)\, dv_2\Big) \Big/ \Big(1 - \varepsilon \int_{\mathbb{R}} w_1(t, x_1, v_2)\, dv_2\Big)$$

[One needs to differentiate expression (4.4) with respect to t and use the Liouville equation for $w_1'(t, y_1, v_1)$]. Equality (4.4) also implies the boundary conditions on the solution of the kinetic equation follow also: for all $v_1 \in \mathbb{R}$,

$$w_1(t, x_1, v_1) = w_1(t, x_1, -v_1) \qquad \text{at } x_1 = 0 \text{ and } x_1 = a. \tag{4.6}$$

The kinetic equation (4.5) differs from the Boltzmann equation in structure. First, it is invertible: the change of variables $t \to -t$ and $v_1 \to -v_1$ does not change its form. Second, if the initial distribution is homogeneous (i.e., does not depend on x_1:

$$w_1(0, x_1, v_1) = f(v_1)$$

and satisfies the boundary conditions (4.6) [$f(-v_1) = f(v_1)$ for any $v_1 \in \mathbb{R}$], it remains unchanged for $t > 0$, whereas the solution of the Boltzmann equation with any homogeneous initial value converges with time to the Maxwell distribution, according to the H-theorem. Third, Eq. (4.5) is applicable for any concentration of spheres ε. The kinetic equation (4.5) can be regarded as a Vlasov type equation because it also has a solution that contains a δ-function. This particular solution is an analog of the hydrodynamic equations.

The kinetic equation (4.5) is obtained in [6]. The one-dimensional model of elastic spheres is generally formulated in [7].

5. Bogolyubov Chain. *Definition 1.* For $N = ms$, a solution of the Liouville equation (in the coordinates y) that has the form

$$w'(t, y, v) = \prod_{i=1}^{m} w_s'(t, y_{(i-1)s+1}, v_{(i-1)s+1}, \ldots, y_{is}, v_{is}) \tag{5.1}$$

will be called s-multiplicative. Its s-particle projection w_s' will be considered symmetric.

The kinetic equation (4.5) is derived in the class of 1-multiplicative functions w'. Actually, it is valid for a broader set of motions.

Theorem 1. *For any $s \geq 1$ in the class of s-multiplicative functions w' (in the coordinates y), the 1-particle projection w_1 (in the coordinates x) in the limit as $N \to \infty$ and fixed s and $\varepsilon = Nd_1$ satisfies the closed kinetic equation* (4.5).

Proof. Function (5.1), generally speaking, is not completely symmetric. Therefore, it needs to be symmetrized, i.e., it is necessary to take the average of

$$\sigma w' = w_s'(t, y_{\sigma(1)}, v_{\sigma(1)}', \ldots, y_{\sigma(s)}, v_{\sigma(s)}') \cdots w_s'(t, y_{\sigma(N-s+1)}, v_{\sigma(N-s+1)}', \ldots, y_{\sigma(N)}, v_{\sigma(N)}')$$

over all permutations $\sigma \in S$. The set $M = \{1, \ldots, N\}$ is divided into subsets M_j, which contain the indices of the variables of the jth factor of the product $\sigma w'$. We assume that $1 \in M_{j_0}$ and β elements from $M_{j_0} \backslash \{1\}$ are located on the left of η [i.e., belong to the set $\{\alpha_1, \ldots, \alpha_{i-1}\}$; see (4.2)], and the remaining elements are on the right. We

denote by l_γ the number of subsets M_j ($j \neq j_0$) that have γ elements on the left of η. Since on the left of η there are $i-1$ numbers, and on the right $N-i$ numbers, the introduced quantities satisfy the equations

$$\sum_{\gamma=1}^{s} \gamma l_\gamma + \beta = i-1, \qquad \sum_{\gamma=0}^{s-1} (s-\gamma) l_\gamma + s - 1 - \beta = N - i. \tag{5.2}$$

The integral of $\sigma w'$ over the set (4.2) is equal to

$$w_1(t, x_1, v_1) = \sum_{i,\beta} \sum_{l_0,\ldots,l_s} w_s'^{(\beta)}(t, \eta, v_1) \binom{s-1}{\beta} \binom{m-1}{l_0 \ldots l_s} \prod_{\gamma=0}^{s} \left(\binom{s}{\gamma} p_\gamma \right)^{l_\gamma}, \tag{5.3}$$

where $w_s'^{(\beta)}$ is obtained from w_s' by the change $y_1 = \eta$, $v_1' = v_1$ and by integration over the remaining variables $y_2, \ldots, y_s$ and $v_2', \ldots, v_s'$ on the set $[0,\eta]^\beta \times [\eta, 1]^{s-1-\beta} \times \mathbb{R}^{s-1}$; similarly, p_γ is obtained by integrating w_s' on the set $[0,\eta]^\gamma \times [\eta,1]^{s-\gamma} \times \mathbb{R}^s$. The expression (5.3) does not depend on the choice of the permutation σ, and, hence, is equal to the average over σ.

In equalities (5.2) for $N \to \infty$, it is possible to discard the number β compared to the remaining terms. Consequently, the index i and the quantity η do not depend on β; therefore, the sum over β is equal to $w_1'(t, \eta, v_1)$. After that, we can retain only the summation over $l_0 \geq 0, \ldots, l_s \geq 0$ ($l_0 + \ldots + l_s = m-1$) expressing i in terms of these indices from the first equation (5.2), where $\beta = 0$. As a result of these transformations, expression (5.3) becomes

$$w_1(t, x_1, v_1) = \sum_{l_0+\ldots+l_s=m-1} w_1'(t, \eta, v_1) \binom{m-1}{l_0 \ldots l_s} \prod_{\gamma=0}^{s} q_\gamma^{l_\gamma}, \tag{5.4}$$

where

$$q_\gamma = \binom{s}{\gamma} p_\gamma \geq 0, \qquad \sum_{\gamma=0}^{s} q_\gamma = 1.$$

We let $N \to \infty$ for fixed s, $z_\gamma = l_\gamma/(m-1)$ ($0 \leq \gamma \leq s$), and $\varepsilon = (N-s)d$. Under these conditions, the sum on the right of (5.4) converges to the integral

$$w_1(t, x_1, v_1) = \int\limits_{z_1+\ldots+z_s \leq 1} w_1'(t, \eta, v_1) \delta(z_1 - q_1, \ldots, z_s - q_s)\, dz_1 \cdots dz_s = w_1'(t, \eta, v_1) J^{-1} \Big|_{z_1=q_1,\ldots,z_s=q_s}.$$

We calculate the Jacobian

$$J = \frac{\partial(z_1 - q_1, \ldots, z_s - q_s)}{\partial(z_1, \ldots, z_s)} = 1 + \varepsilon \int\limits_{\mathbb{R}} w_1'(t, \eta, v_2)\, dv_2.$$

At the point $z_1 = q_1, \ldots, z_s = q_s$, the following equality holds:

$$\eta = x_1 - \varepsilon s^{-1} \sum_{\gamma=1}^{s} \gamma q_\gamma = x_1 - \varepsilon \int\limits_0^\eta \int\limits_{\mathbb{R}} w_1'(t, y_2, v_2)\, dy_2\, dv_2.$$

Thus, we obtain formula (4.4), quod erat demonstrandum.

Theorem 2. *For any $s \leq s'$ in the class of symmetrized s'-multiplicative functions w' (in the coordinates y), the s-particle projection w_s (in the coordinates x) in the limit as $N \to \infty$ has the form $\prod_{i=1}^{s} w_1(t, x_i, v_i)$, where $w_1(t, x_1, v_1)$ is expressed in terms of $w_1'(t, y_1, v_1)$ by formula* (4.4).

The proof is similar to the proof of Theorem 1. The difference is as follows. The distinguished pairs of variables y_i and v_i' ($1 \leq i \leq s$) now can be in different factors of the product $\sigma w'$, depending on the permutation σ. The fraction of those permutations for which each factor contains not more than one distinguished pair tends to 1 as $N \to \infty$. Therefore, in the limit, the s-particle projection w_s is itself a 1-multiplicative one.

To obtain the multidimensional generalization of the kinetic equation (4.5), it is necessary to consider unsymmtrized multiplicative solutions w' of the Liouville equations in the coordinates y. In the initial phase space

Φ_x, they specify multivalued solutions, whose branches are sewed together on the boundary in such a way that the uniqueness theorem for the phase trajectories remains valid.

Theorem 3. *For any $s \le s'$ in the class unsymmetrized s'-multiplicative functions w' (in the coordinates y), the s-particle projection w_s (in the coordinates x) in the limit as $N \to \infty$ has the form*

$$w_s(t, x_1, v_1, \ldots, x_s, v_s) = w'_s(t, y_1, v_1, \ldots, y_s, v_s) \prod_{i=1}^{s} \Big(1 + \varepsilon \int\limits_{\mathbb{R}} w'_1(t, y_i, v_{s+1})\, dv_{s+1}\Big)^{-1},$$

where

$$x_i = y_i + \varepsilon \int\limits_0^{y_i} \int\limits_{\mathbb{R}} w'_1(t, y_{s+1}, v_{s+1})\, dy_{s+1}\, dv_{s+1} \qquad (i = 1, \ldots, s),$$

the function w'_s satisfies the Liouville equation (in the coordinates y) and w'_1 is its 1-particle projection.

According to Theorem 2, all s-particle projections are uniquely expressed in terms of the 1-particle projection. Thus, for one-dimensional motion in the class of multiplicative distributions, for which the kinetic equation is valid, the Bogolyubov assumption is satisfied.

The question arises: Is the kinetic equation (4.5) valid in the class of arbitrary distributions? No it is not. Indeed, we consider the half-sum of two 1-multiplicative solutions of the Liouville equation in the coordinates y. In the limit as $N \to \infty$, it corresponds to the one-particle projection w_1, which is the half-sum of two solutions of Eq. (4.5). Since this equation is nonlinear, the function w_1 does not satisfy it. The function w_1 cannot be a solution of any closed kinetic equation because it is no uniquely determined by the initial value at $t = 0$.

Does the kinetic equation exist in the class of arbitrary distributions in the case of low concentrations of the spheres ε? The Boltzmann equation is valid only for diluted gases. To answer this question, we calculate the integral of an arbitrary function w' over the set (4.2) by expansion in a series of powers of d_1 for fixed N. We obtain

$$w_1(t, x_1, v_1) = w'_1(t, x_1, v_1)$$

$$- \varepsilon \int\limits_{\mathbb{R}} \Big(w'_2(t, x_1, v_1, x_1, v_2) + \int\limits_0^{x_1} \frac{\partial w'_2}{\partial y_1}(t, x_1, v_1, y_2, v_2)\, dy_2\Big) dv_2 + O(\varepsilon^2). \tag{5.5}$$

From this it follows that in the class of arbitrary distributions, the Bogolyubov chain is not closed as a first order of accuracy in ε. In the class of symmetrized multiplicative functions, the projection w'_2 (in the limit as $N \to \infty$) is uniquely expressed in terms of w'_1, and in this case, expansion (5.5) is also obtained from formula (4.4).

Without giving a proof, we note that the kinetic equation (4.5) can also be obtained by heuristic reasoning similar to that used in the theory of gases. Then, the class of multiplicative distributions does not appear explicitly but there is the illusion that the kinetic equation is valid in the class of arbitrary distributions.

6. Equations of Hydrodynamics. An analog of the equations of multivelocity hydrodynamics (plasma) is obtained as a particular solution of the kinetic equation (4.5) in the form

$$w(t, x, v) = \sum_{i=1}^{k} \rho_i(t, x) \delta(v - u_i(t, x)) \tag{6.1}$$

(the subscript 1 is omitted). Setting the factors of the δ-function and its derivative on the δ-function support equal to zero, we first satisfy the kinetic equation, and second, obtain a closed system of equations for the unknown functions ρ_i and u_i $(i = 1, \ldots, k)$. For example, for $k = 2$, we obtain the following system of hydrodynamic equations:

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial\,(\rho_i u_i)}{\partial x} = 0, \qquad \frac{\partial u_i}{\partial t} + \frac{u_i - \varepsilon(\rho_1 u_1 + \rho_2 u_2)}{1 - \varepsilon(\rho_1 + \rho_2)} \frac{\partial u_i}{\partial x} = 0 \qquad (i = 1, 2). \tag{6.2}$$

The boundary conditions (4.6) for the kinetic equation entails different boundary conditions for system (6.2), in particular, $\rho_1 = \rho_2$ and $u_1 = -u_2$ for $x = 0$ and $x = a$.

The physical meaning of Eqs. (6.2) is as follows. In the vicinity of each point x, there are two sorts of spheres that move at velocities u_1 and u_2 and have concentrations ρ_1 and ρ_2, respectively. The temperature and pressure

of this one-dimensional gas can be determined in the same way as for a three-dimensional gas. Then, the similarity between Eqs. (6.2) and the gas-dynamic equations becomes even more noticeable. However, since a one-dimensional gas does not exist in nature, this mechanical analogy is hardly of great scientific interest. The system of equations of the form (6.2) is of significance for the theory of partial equations as heuristic examples.

The problem (4.5), (4.6) is generally solvable in time. Indeed, if

$$h(0,x) = \int\limits_{\mathbb{R}} w(0,x,v)\,dv < \varepsilon^{-1}, \tag{6.3}$$

then formula (4.4) uniquely defines the initial function $w'(0,y,v)$ and, hence, the solution $w'(t,y,v)$ of the Liouville equation in the coordinate system y that is defined for all $t \geq 0$; the solution $w(t,x,v)$ of the kinetic equation (4.5) is expressed in terms of the latter solution. Inequality (6.3) is satisfied in the sense of the one-particle projection. The sum of the diameters of the spheres located on a small interval of length Δx of the axis x, which is equal to $d_1 N h \Delta x = \varepsilon h \Delta x$, should not exceed Δx since the spheres are absolutely rigid. This implies that $\varepsilon h \leq 1$. However, collisions of higher order than paired collision are eliminated from the consideration, and, hence, $\varepsilon h < 1$.

However, the solution of the hydrodynamic equations (6.2) can cease to exist at some time t because of violation of the uniqueness of the functions $u_i(t,x)$. In this case, the function (6.1) as a solution of the kinetic equation retains its meaning and continues to exist for all $t \geq 0$.

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